M-fractional solitons and periodic wave solutions to the Hirota–Maccari system

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Received 25 October 2018
Revised 3 December 2018
Accepted 5 December 2018
Published

In this study, we construct several wave solutions to the nonlinear fractional Hirota–Maccari equation with a truncated M-fractional derivative via the extended sinh-Gordon equation expansion method. The constraint conditions that guarantee the existence of valid solutions are stated. We use suitable values of parameters in plotting the 2- and 3-dimensional graphs of the reported solutions.

Keywords: M-fractional derivative; Hirota–Maccari equation; soliton.

1. Introduction

Nonlinear partial differential equations (NPDEs) are widely used in modeling several complex nonlinear physical phenomena in different fields of nonlinear science such as optical fibers, hydrodynamics, complex acoustics, quantum hall effect, heat pulses in solids and many other nonlinear unstable aspects.1,2 Several computational approaches for constructing various solitons and other solutions to different kind of NPDEs have been reported in the literature such as the sine-Gordon expansion method (ShGEEM),3–5 the first integral method,6,7 the improved Bernoulli sub-equation function method,8,9 the trial solution method,10,11 the new auxiliary equation method,12 the extended simple equation method,13 the solitary wave ansatz method,14 the functional variable method,15 the modified exp(−Φ(η))-expansion function method,16–19 the sub-equation method,20–22 and several others.23–43
However, in this study, the extended ShGEEM\textsuperscript{44–49} is utilized in constructing family of solitons and other solutions to the fractional Hirota–Maccari system\textsuperscript{50} with a truncated M-fractional derivative.

The fractional Hirota–Maccari system with a truncated M-fractional derivative is given as

\begin{equation}
\begin{aligned}
&iD_{M,t}^{\alpha,\beta} \psi + D_{M,y}^{\alpha,\beta} D_{M,x}^{\alpha,\beta} \psi + iD_{M,x}^{3\alpha,\beta} \psi + \psi v - i|\psi|^2 D_{M,x}^{\alpha,\beta} \psi = 0, \\
&3D_{M,x}^{\alpha,\beta} v + D_{M,y}^{\alpha,\beta} (|\psi|^2) = 0,
\end{aligned}
\end{equation}

where \( \psi(x,y,t) \) and \( \psi(x,y,t) \) are complex-valued and real-valued functions, respectively. The variables “\( t \)” and “\( x,y \)” represent the temporal and spatial variables, respectively.

For the past two decades, the field of fractional calculus has attracted the attention of many researchers. Fractional order partial differential equations serve as the generalization of the classical integer order partial differential equations. They are used to model several nonlinear aspects such as biological processes, fluid mechanics, chemical processes and so on.\textsuperscript{51} There are several definitions of fractional derivatives available in the literature, such as the Riemann–Liouville, Caputo and Grunwald–Letnikov definitions, Atangana–Baleanu derivative in Caputo sense, Atangana–Baleanu fractional derivative in Riemann–Liouville sense,\textsuperscript{52,53} the conformable fractional derivative,\textsuperscript{54} Atangana et al.\textsuperscript{55} presented some new properties to the conformable fractional derivative. Recently, Sousa and Oliveira developed the new truncated M-fractional derivative.\textsuperscript{56} This new fractional derivative generalizes the conformable derivative proposed by Khalil et al.\textsuperscript{54}

2. The Truncated M-Fractional Derivative

In this section, some basic definition and theorem about the new truncated M-fractional derivative are given.\textsuperscript{56}

**Definition 1.** Let \( h : [0, \infty) \to \mathbb{R} \), then the new truncated M-fractional derivative of \( h \) of order \( \alpha \) is defined as

\[
D_{M}^{\alpha,\beta} \{(h(t))\} = \lim_{\epsilon \to 0} \frac{h(t/E_{\beta}(\epsilon^{1-\alpha})) - h(t)}{\epsilon}, \quad \forall t > 0, \quad 0 < \alpha < 1, \quad \beta > 0, \tag{2.1}
\]

where \( E_{\beta}(\cdot) \) is a truncated Mittag–Leffler function of one parameter.\textsuperscript{56}

**Theorem 1.** Let \( 0 < \alpha \leq 1, \beta > 0, q, r \in \mathbb{R} \), and \( g, h \) \( \alpha \)-differentiable at a point \( t > 0 \), then:

\[1\] \( D_{M}^{\alpha,\beta} \{(rq + rh)(t)\} = qD_{M}^{\alpha,\beta} \{g(t)\} + rD_{M}^{\alpha,\beta} \{h(t)\} \).

\[2\] \( D_{M}^{\alpha,\beta} \{(g,h)(t)\} = g(t)D_{M}^{\alpha,\beta} \{h(t)\} + h(t)D_{M}^{\alpha,\beta} \{g(t)\} \).

\[3\] \( D_{M}^{\alpha,\beta} \{ \frac{g(t)}{n(t)} \} = \frac{h(t)D_{M}^{\alpha,\beta} \{g(t)\} - g(t)D_{M}^{\alpha,\beta} \{h(t)\}}{h(t)} \).

\[4\] \( D_{M}^{\alpha,\beta} \{c\} = 0 \), where \( g(t) = c \) is a constant.

\[5\] If \( g \) is differentiable, then \( D_{M}^{\alpha,\beta} \{g(t)\} = \frac{t^{1-\alpha}}{\Gamma(\beta+1)} \frac{dg(t)}{dt} \).
3. The Extended ShGEEM

In this section, the steps of the extended (ShGEEM) method are presented.

**Step-1:** Consider the following nonlinear fractional partial differential equation with the new truncated M-fractional derivative:

\[ P(D_{\alpha,\beta}^{\alpha,\beta} \psi, D_{\alpha,\beta}^{\alpha,\beta} \psi, D_{\alpha,\beta}^{\alpha,\beta} \psi, D_{\alpha,\beta}^{\alpha,\beta} \psi, \ldots) = 0. \]  
(3.1)

Putting the fractional traveling wave transformation

\[ \psi(x,t) = \Phi(\zeta), \quad \zeta = \Gamma(\beta + 1)\alpha \nu (x^\alpha - vt^\alpha), \]  
(3.2)

into Eq. (3.1), produces the following nonlinear ordinary differential equation (NODE):

\[ D(\Phi, \Phi', \Phi'', \Phi^2 \Phi', \ldots) = 0, \]  
(3.3)

**Step-2:** We suppose the trial solution to Eq. (3.3) to be of the form

\[ \Phi(\Theta) = \sum_{k=1}^{m} [B_k \sinh(\Theta) + A_k \cosh(\Theta)]^k + A_0, \]  
(3.4)

where \( A_0, A_k, B_k \) (\( k = 1, 2, \ldots, m \)) are constants to be determined later and \( \Theta \) is a function of \( \zeta \) which satisfies the following ordinary differential equations:

\[ \Theta' = \sinh(\Theta), \]  
(3.5)

\[ \Theta' = \cosh(\Theta). \]  
(3.6)

The value of \( m \) is determined by using the homogeneous balance principle.

Equations (3.5) and (3.6) are obtained from the fractional sinh-Gordon equation given by

\[ D_{\alpha,\beta}^{\alpha,\beta} \psi = \gamma \sinh(\psi). \]  
(3.7)

Equations (3.5) and (3.6) pose the following solutions:

\[ \sinh(\Theta) = \pm \cosh(\zeta) \quad \text{or} \quad \sinh(\Theta) = \pm i \sech(\zeta), \]  
(3.8)

\[ \cosh(\Theta) = \pm \coth(\zeta) \quad \text{or} \quad \cosh(\Theta) = \pm \tanh(\zeta) \]  
(3.9)

and

\[ \sinh(\Theta) = \tan(\zeta) \quad \text{or} \quad \sinh(\Theta) = -\cot(\zeta), \]  
(3.10)

\[ \cosh(\Theta) = \pm \sec(\zeta) \quad \text{or} \quad \cosh(\Theta) = \pm \csc(\zeta), \]  
(3.11)

respectively, where \( i = \sqrt{-1}. \)

**Step-3:** Putting Eq. (3.4), its possible derivatives with the fixed value of \( m \) along with Eq. (3.5) and/or Eq. (3.6) into Eq. (3.3), yields an equation in powers of hyperbolic functions; \( \Theta^l \sinh'(\Theta) \cosh'(\Theta) \) \((l = 0, 1 \text{ and } i, j = 0, 1, 2, \ldots)\). We collect a set of overdetermined nonlinear algebraic equations in \( A_0, A_k, B_k, \nu, \nu \) by setting the coefficients of \( \Theta^l \sinh'(\Theta) \cosh'(\Theta) \) to zero.
Step-4: The collected set of overdetermined nonlinear algebraic equations is then solved with the aid of computational software to determine the values of the parameters $A_0, A_k, B_k, \nu, v$.

Step-5: Based on Eqs. (3.8)–(3.10) and (3.11), Eq. (3.1) poses the following forms of solutions:

\[
\Phi(\zeta) = \sum_{k=1}^{m} \left[ \pm B_k \text{sech}(\zeta) \pm A_k \tanh(\zeta) \right]^k + A_0, \quad (3.12)
\]

\[
\Phi(\zeta) = \sum_{k=1}^{m} \left[ \pm B_k \text{csch}(\zeta) \pm A_k \coth(\zeta) \right]^k + A_0, \quad (3.13)
\]

\[
\Phi(\zeta) = \sum_{k=1}^{m} \left[ \pm B_k \sec(\zeta) + A_k \tan(\zeta) \right]^k + A_0 \quad (3.14)
\]

and

\[
\Phi(\zeta) = \sum_{k=1}^{m} \left[ \pm B_k \csc(\zeta) - A_k \cot(\zeta) \right]^k + A_0. \quad (3.15)
\]

4. Application

In this section, we present the application of the ShGEEM to the Hirota–Maccari system.

Consider equation (Eq. (1.1)) given in Sec. 1. Inserting the fractional wave transformation

\[
\psi(x, y, t) = \Phi(\zeta)e^{i\theta}, \quad v(x, y, t) = V(\zeta), \quad \zeta = \frac{\Gamma(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - rt^\alpha),
\]

\[
\theta = \frac{\Gamma(\beta + 1)}{\alpha} (ax^\alpha + by^\alpha + rt^\alpha),
\]

into Eq. (1.1), we get the following system of NODE and the constraint condition:

\[
(3a - 1)\Phi^3 + 3(a^3 - ab - r)\Phi + 3\mu^2(1 - 3a)\Phi'' = 0, \quad V + \frac{\Phi^2}{3} = 0 \quad (4.1)
\]

and

\[
\kappa = (a + b - 3a^2) - \frac{(a^3 - ab - r)}{(1 - 3a)}, \quad (4.2)
\]

respectively.

Balancing the terms $\Phi^3$ and $\Phi''$, yields $m = 1$.

With $m = 1$, Eq. (3.4) takes the form

\[
\Phi(\Theta) = B_1 \sinh(\Theta) + A_1 \cosh(\Theta) + A_0. \quad (4.3)
\]

Substituting Eq. (4.3) and its second derivative along with Eq. (3.5) and/or (3.6), gives an equation in powers of hyperbolic functions. We collect the set of overdetermined nonlinear algebraic equations as explained in the description of the method.
We further simplify the set of algebraic equations to obtain the values of the parameters. To get the solutions of Eq. (1.1), we substitute the values of the parameters into Eqs. (3.12)–(3.15).

Case-1: When we get

\[ \psi_1(x, y, t) = \pm \sqrt{\frac{ab - a^3 + r}{a - \frac{r}{3}}} \left( i \text{sech} \left[ \frac{\Gamma(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - kt^\alpha) \right] \right. \]

\[ + \left. \tanh \left[ \frac{\Gamma(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - kt^\alpha) \right] e^{i \left( \frac{\Gamma(\beta + 1)}{\alpha} \alpha + by^\alpha + rt^\alpha \right)} \right), \]  

(4.4)

\[ \psi_2(x, y, t) = \pm \sqrt{\frac{ab - a^3 + r}{a - \frac{r}{3}}} \left( \text{coth} \left[ \frac{\Gamma(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - kt^\alpha) \right] \right. \]

\[ + \left. \text{csch} \left[ \frac{\Gamma(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - kt^\alpha) \right] e^{i \left( \frac{\Gamma(\beta + 1)}{\alpha} \alpha + by^\alpha + rt^\alpha \right)} \right), \]  

(4.6)

where \((a - \frac{r}{3})(ab - a^3 + r) > 0, (3a - 1)(ab - a^3 + r) > 0\) for valid solitons.

Case-2: When we get

\[ \psi_3(x, y, t) = \pm \sqrt{\frac{ab - a^3 + r}{a - \frac{r}{3}}} \tan \left[ \frac{\Gamma(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - kt^\alpha) \right. \]

\[ \times \left. e^{i \left( \frac{\Gamma(\beta + 1)}{\alpha} \alpha + by^\alpha + rt^\alpha \right)} \right), \]  

(4.8)

\[ \psi_4(x, y, t) = \frac{(ab - a^3 + r)}{(3a - 1)} \tan^2 \left[ \frac{\Gamma(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - kt^\alpha) \right] \]  

(4.9)
and

\[ \psi_4(x, y, t) = \pm \sqrt{a - a^3 + r \over 3a - 1} \cosh \left[ \Gamma(\beta + 1) \alpha x^\alpha + y^\alpha - \kappa t^\alpha \right] \times e^{i \left[ \Gamma(\beta + 1) \alpha (ax^\alpha + by^\alpha + rt^\alpha) \right]}, \]

(4.10)

\[ v_4(x, y, t) = -\left( \frac{ab - a^3 + r}{3a - 1} \right) \cosh^2 \left[ \Gamma(\beta + 1) \alpha x^\alpha + y^\alpha - \kappa t^\alpha \right], \]

(4.11)

where \((a - a^3)/r > 0, (6a - 2)(ab - a^3 + r) > 0\) for valid solitons.

**Case-3:** When

\[ A_0 = 0, \quad A_1 = 0, \quad B_1 = -\sqrt{6(a^3 - ab - r) \over 3a - 1}, \quad \mu = -\sqrt{ab + r - a^3 \over 1 - 3a}, \]

we get

\[ \psi_5(x, y, t) = \pm \sqrt{6(a^3 - ab - r) \over 3a - 1} \sech \left[ \Gamma(\beta + 1) \alpha x^\alpha + y^\alpha - \kappa t^\alpha \right] \times e^{i \left[ \Gamma(\beta + 1) \alpha (ax^\alpha + by^\alpha + rt^\alpha) \right]}, \]

(4.12)

\[ v_5(x, y, t) = -\frac{2(ab + r - a^3)}{(3a - 1)} \sech^2 \left[ \Gamma(\beta + 1) \alpha x^\alpha + y^\alpha - \kappa t^\alpha \right], \]

(4.13)

and

\[ \psi_6(x, y, t) = \pm \sqrt{6(a^3 - ab - r) \over 3a - 1} \csch \left[ \Gamma(\beta + 1) \alpha x^\alpha + y^\alpha - \kappa t^\alpha \right] \times e^{i \left[ \Gamma(\beta + 1) \alpha (ax^\alpha + by^\alpha + rt^\alpha) \right]}, \]

(4.14)

\[ v_6(x, y, t) = -\frac{2(a^3 - ab - r)}{(3a - 1)} \csch^2 \left[ \Gamma(\beta + 1) \alpha x^\alpha + y^\alpha - \kappa t^\alpha \right], \]

(4.15)

where \((1 - 3a)(ab + r - a^3) > 0\) for valid solitons.

**Case-4:** When

\[ A_0 = 0, \quad A_1 = \sqrt{a^3 - ab - r \over a - 3} \cdot B_1 = A_1, \quad \mu = -\sqrt{2(a^3 - ab - r) \over 3a - 1}, \]

we get

\[ \psi_7(x, y, t) = \pm \sqrt{a^3 - ab - r \over a - 3} \left( \sec \left[ \Gamma(\beta + 1) \alpha x^\alpha + y^\alpha - \kappa t^\alpha \right] \right) + \tan \left[ \Gamma(\beta + 1) \alpha x^\alpha + y^\alpha - \kappa t^\alpha \right] \times e^{i \left[ \Gamma(\beta + 1) \alpha (ax^\alpha + by^\alpha + rt^\alpha) \right]}, \]

(4.16)
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\[ v_7(x, y, t) = \left( a^3 - ab - r \right) \left( \frac{3a - 1}{a} \right) \left( \sec \left[ \frac{(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - \kappa t^\alpha) \right] + \tan \left[ \frac{(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - \kappa t^\alpha) \right] \right)^2, \]  
\[ \psi_8(x, y, t) = \pm \sqrt{\frac{3a - 1}{a} \left( \cot \left[ \frac{(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - \kappa t^\alpha) \right] + \csc \left[ \frac{(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - \kappa t^\alpha) \right] \right)} e^{i \frac{(\beta + 1)}{\alpha} (ax^\alpha + by^\alpha + rt^\alpha)}, \]  
\[ v_8(x, y, t) = -\frac{a^3 - ab - r}{3a - 1} \left( \cot \left[ \frac{(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - \kappa t^\alpha) \right] + \csc \left[ \frac{(\beta + 1)}{\alpha} \mu(x^\alpha + y^\alpha - \kappa t^\alpha) \right] \right)^2, \]

where \((3a - 1)(a^3 - ab - r) > 0\) for valid solutions.

5. Results and Discussion

This section discusses the reported results in this study. Using the powerful extended ShGEEM, several solitary wave solutions such as the topological, non-topological, mixed topologiça-nontopological, singular solitons and singular periodic wave solutions are successfully constructed.

In order to have clear and good understanding of the physical properties of the constructed topological, nontopological, mixed topologica-nontopological, singular solitons and singular periodic wave solutions, under the choice of the suitable values of parameters and the fractional value of \(\alpha\), the 3D and 2D graphs are plotted.

Fig. 1. The (a) 3D and (b) 2D plots of the topological soliton (Eq. (4.8)) at \(a = 2, \kappa = 0.25, b = 4, r = 0.35, \alpha = 0.95, \beta = 0.45, t = 0.5, \) and \(y = 0.65\) for the 2D graph.
Fig. 2. The (a) 3D and (b) 2D plots of the topological soliton (Eq. (4.9)) at $a = 2$, $\kappa = 0.25$, $b = 4$, $r = 0.35$, $\alpha = 0.95$, $\beta = 0.45$, $t = 0.5$, and $y = 0.65$ for the 2D graph.

Fig. 3. The (a) 3D and (b) 2D plots of the singular soliton (Eq. (4.10)) at $a = 2$, $\kappa = 0.25$, $b = 4$, $r = 0.35$, $\alpha = 0.95$, $\beta = 0.45$, $t = 0.5$, and $y = 0.65$ for the 2D graph.

The perspective view of the topological solitons (Eqs. (4.8) and (4.9)) can be seen in the 3D graphs which appear in the (a) parts of Figs. 1 and 2. The perspective view of the singular solitons (Eqs. (4.10) and (4.11)) can be seen in the 3D graphs which appear in the (a) parts of Figs. 3 and 4. The perspective view of the nontopological solitons (Eqs. (4.12) and (4.13)) can be seen in the 3D graphs which appear in the (a) parts of Figs. 5 and 6. The perspective view of the topological solitons (Eqs. (4.16) and (4.17)) can be seen in the 3D graphs which appear in the (a) parts of Figs. 7 and 8. The propagation pattern of the wave of the studied nonlinear model with respect to the constructed solutions along the $x$-axis are illustrated in the 2D graphs which appear at the (b) parts of Figs. 1–8.
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Fig. 4. The (a) 3D and (b) 2D plots of the singular soliton (Eq. (4.11)) at $a = 2$, $\kappa = 0.25$, $b = 4$, $r = 0.35$, $\alpha = 0.95$, $\beta = 0.45$, $t = 0.5$, and $y = 0.65$ for the 2D graph.

Fig. 5. The (a) 3D and (b) 2D plots of the nontopological soliton (Eq. (4.12)) at $a = -2$, $\kappa = 0.25$, $b = 4$, $r = 0.35$, $\alpha = 0.95$, $\beta = 0.45$, $t = 0.5$, and $y = 0.65$ for the 2D graph.

Fig. 6. The (a) 3D and (b) 2D plots of the nontopological soliton (Eq. (4.13)) at $a = -2$, $\kappa = 0.25$, $b = 4$, $r = 0.35$, $\alpha = 0.95$, $\beta = 0.45$, $t = 0.5$, and $y = 0.65$ for the 2D graph.
6. Conclusions

In this study, several wave solutions to the fractional Hirota–Maccari system with a truncated M-fractional derivative are successfully revealed by using the extended ShGEEM. The truncated M-fractional derivative is a generalized form of the conformable fractional derivative. The definition of the new fractional derivative is smoothly used in transforming the fractional Hirota–Maccari system to nonlinear ordinary differential equation. The reported results may be useful in explaining the physical meaning of the studied nonlinear model. The extended ShGEEM is a powerful mathematical technique which can be used in obtaining several wave solutions to various complex fractional nonlinear mathematica models.
References

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